A Bayesian nonparametric approach for causal inference with multiple mediators

Samrat Roy, Michael J. Daniels, Brendan J. Kelly, Jason Roy

UPenn, UF, UPenn, Rutgers

Perelman School of Medicine, University of Pennsylvania

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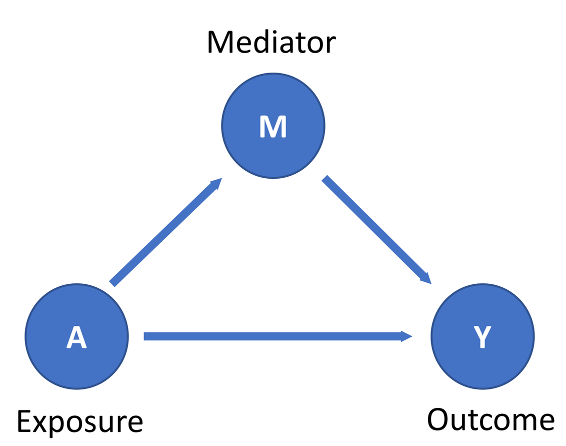
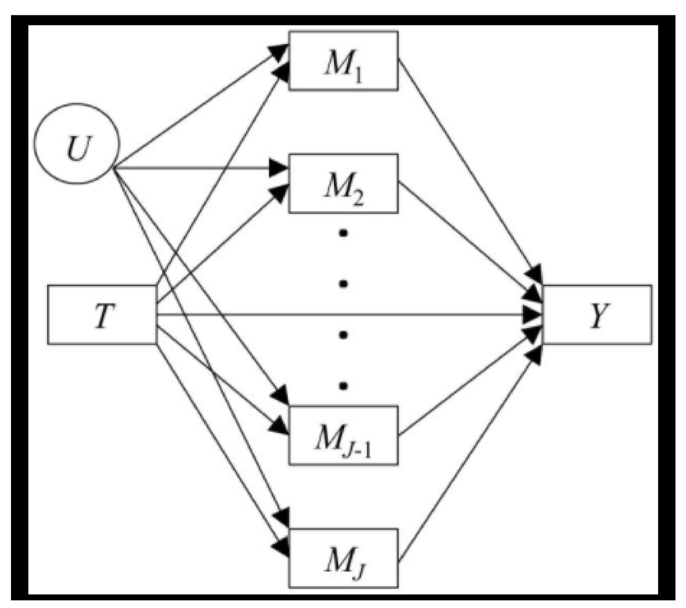
Story so far....





2013, Wang et al. →

$$\text{logit}(\Pr(Y=1)) = \beta_0 + \beta_1 M_1 + \ldots + \beta_J M_J + \beta_{J+1} T + \boldsymbol{\beta}' \boldsymbol{W}$$
$$M_j^{(*)} = \alpha_{j0} + \alpha_{j1} T + \boldsymbol{\alpha_j}' \boldsymbol{W} + \varepsilon_j, j = 1, \ldots, J.$$
$$IE = IE_1 + IE_2 + , \ldots, + IE_{J-1} + IE_J$$

2013, Imai and Yamamoto →

$$M_i = \alpha_M + \beta_M T_i + \xi_M^\top X_i + \varepsilon_{iM},$$
$$W_i = \alpha_W + \beta_W T_i + \xi_W^\top X_i + \varepsilon_{iW},$$
$$Y_i = \alpha_3 + \beta_3 T_i + \gamma M_i + \theta^\top W_i + \xi_3^\top X_i + \varepsilon_{i3}.$$

2014, VanderWeele and Vansteelandt →

$$E[Y \mid a, \mathbf{m}, c] = \theta_0 + \theta_1 a + \theta_2^{(1)} m(1) + \theta_2^{(2)} m^{(2)} + \ldots + \theta_2^{(K)} m^{(K)} + \theta_4' c$$
$$E[M^{(i)} \mid a, c] = \beta_0^{(i)} + \beta_1^{(i)} a + \beta_2^{(i)'} \text{ for } i = 1, \ldots, K.$$

2019, Sohn et al. →

$$\boldsymbol{M}_i = \left(\boldsymbol{m}_0 \oplus \boldsymbol{a}^{T_i} \oplus \boldsymbol{h}^{X_i}\right) \oplus \boldsymbol{U}_{1i}$$
$$Y_i = c_0 + cT_i + (\log \boldsymbol{M}_i)^\top \boldsymbol{b} + gX_i + U_{2i}, \text{ subject to } \boldsymbol{b}^\top \mathbf{1}_k = 0,$$

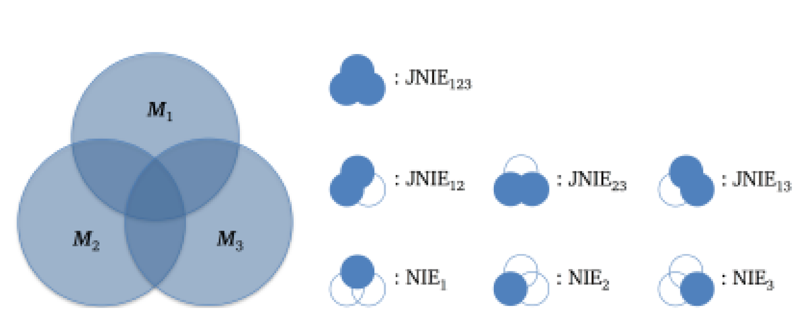
These approaches suffer from **parametric misspecification** and assume that the joint mediation effect is the **sum** of the individual mediator's effects

What we want to do?

- A flexible approach that **does not** suffer from parametric misspecification
- Estimate **all possible interaction effects** of the mediators instead of just the additive joint mediation effect

Causal Effects and Identification

$$TE = E[Y(1, M(1,1,\cdots,1)) - Y(0, M(0,0,\cdots,0))]$$
$$NDE = E[Y(1, M(0,0,\cdots,0)) - Y(0, M(0,0,\cdots,0))]$$
$$JNIE = TE - NDE = E[Y(1, M(1,1,\cdots,1)) - Y(1, M(0,0,\cdots,0))]$$
$$JNIE_1 = E[Y(1, M(1,1,1)) - Y(1, M(0,1,1))]$$
$$JNIE_{12} = E[Y(1, M(1,1,1)) - Y(1, M(0,0,1))]$$



Causal Effects and Identification (Cont.)

Assumption 1 $\{Y(a, M(a,a,\cdots,a)), M(0,0,\cdots,0), M(1,1,\cdots,1)\} \perp\!\!\!\perp A | L = \ell\}$.

Assumption 2 *For exposure $A = 1$, the conditional distributions of the observable potential outcome $Y(1, M(1,1,\cdots,1))$ given values of all potential mediators (and confounders), is the same as that of a priori counterfactual $Y(1, M(0,0,\cdots,0))$, regardless of whether the mediator values were induced by $A = 1$ or $A = 0$.*

the above assumption implies that,

$$f_{1,M(0,0,\cdots,0)}(y \mid M(0,0,\cdots,0) = m, L = \ell)$$
$$= f_{1,M(1,1,\cdots,1)}(y \mid M(1,1,\cdots,1) = m, L = \ell).$$

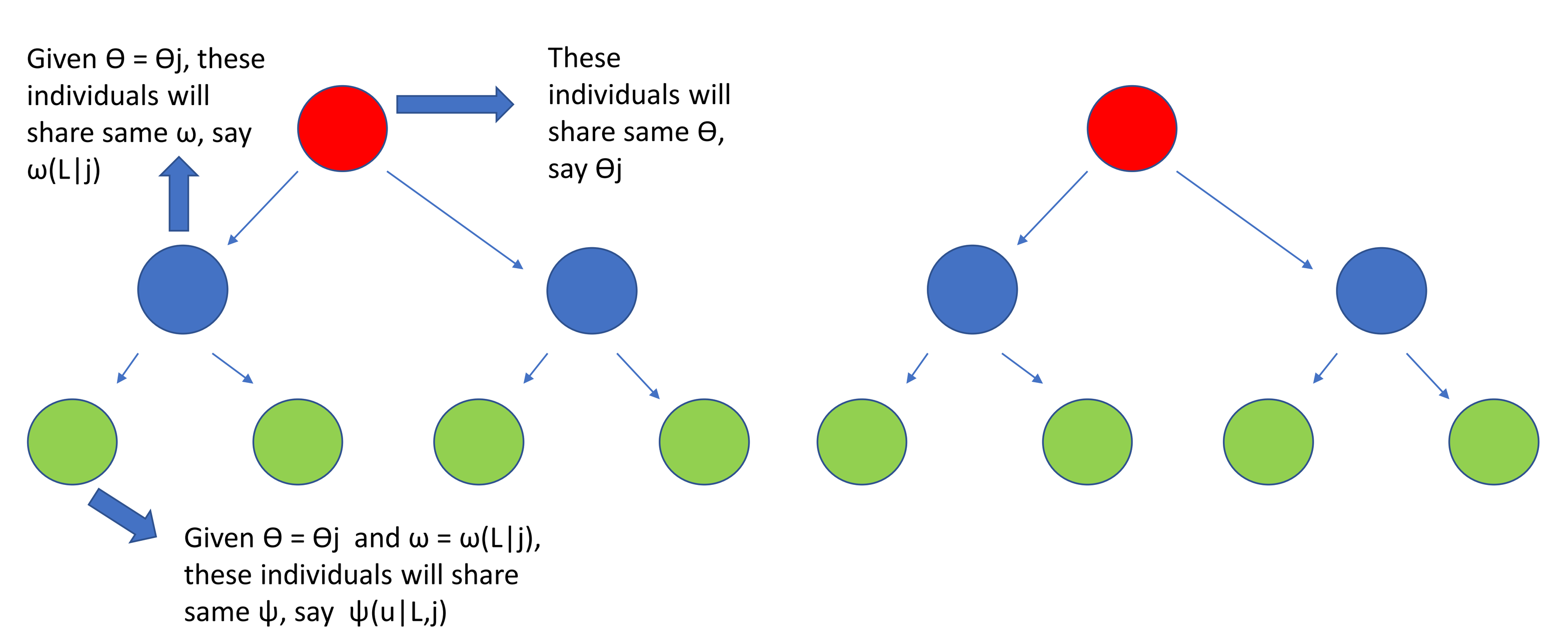
Assumption 3 $M_{j_1}(a) \perp\!\!\!\perp M_{j_2}(a') | L$ *for* $a \neq a'$ *and* $j_1, j_2 = 1, 2, \cdots, Q$

Theorem 1 *Under the above assumptions, the NDE, JNIE (and its various decompositions) are identifiable.*

BNP Model for the observed data

$$Y_i | M_i, X_i; \theta_i \sim p(y|m, x; \theta_i)$$
$$M_{iq} | X_i; \omega_i \sim p(m_q | x; \omega_i) : q = 1, \ldots, Q$$
$$X_{i,r} | \psi_i \sim p(x_r | \psi_i) : r = 1, \ldots, p+1,$$
$$(\theta_i, \omega_i, \psi_i) | P \sim P, \quad P \sim EDP3(\alpha_\theta, \alpha_\omega, \alpha_\psi, P_0).$$

$P \sim EDP3(\alpha_\theta, \alpha_\omega, \alpha_\psi, P_0)$ means that $P_\theta \sim DP(\alpha_\theta, P_{0,\theta})$, $P_{\omega|\theta} \sim DP(\alpha_\omega, P_{0,\omega|\theta})$, and $P_{\psi|\theta,\omega} \sim DP(\alpha_\psi, P_{0,\psi|\theta,\omega})$ with base measure $P_0 = P_{0,\theta} \times P_{0,\omega|\theta} \times P_{0,\psi|\theta,\omega}$. (Enriched Dirichlet Process).



- Assume (local / within cluster) generalized linear models for $Y_i | M_i, X_i; \theta_i$ and $M_{iq} | X_i$: $K(y_i | m_i, x_i; \theta_j)$ and $K(m_i | x_i; \omega_{l|j})$
- Given ψ_i (that means a particular third level cluster), the $(p+1)$ covariates, $X_{i,1}, X_{i,2}, \cdots, X_{i,(p+1)}$ are assumed to be *"locally"* (that is, intra-cluster) independent
- Similarly, given ω_i (that means a particular second level cluster) and conditional on the covariates X_i, the Q mediators $M_{i1}, M_{i2}, \cdots, M_{iQ}$ are assumed to be "locally" independent

Question: What happens "globally" ?

BNP Model (Cont.)

- Joint distribution:

$$f(y_i, m_i, x_i | P) = \sum_{j=1}^{\infty} \pi_j \, K(y_i | m_i, x_i; \theta_j) \sum_{l=1}^{\infty} \pi_{l|j} \, K(m_i | x_i; \omega_{l|j}) \sum_{u=1}^{\infty} \pi_{u|j,l} \, K(x_i | \psi_{u|j,l}).$$

where $\pi_j = \pi'_j \prod_{j'<j}(1 - \pi'_{j'})$, $\pi_{l|j} = \pi'_{l|j} \prod_{l'<l}(1 - \pi'_{l'|j})$ and $\pi_{u|j,l} = \pi'_{u|j,l} \prod_{u'<u}(1 - \pi'_{u'|j,l})$ with $\pi'_j \sim \text{Beta}(1, \alpha_\theta)$, $\pi'_{l|j} \sim \text{Beta}(1, \alpha_\omega)$ and $\pi'_{u|j,l} \sim \text{Beta}(1, \alpha_\psi)$

- Conditional distribution of Y given M and X:

$$f(y|m, x) = \sum_{j=1}^{\infty} W_j(m, x) \cdot K(y|m, x; \theta_j)$$

where,

$$W_j(m, x) = \frac{\pi_j \sum_{l=1}^{\infty} \pi_{l|j} \, K(m|x; \omega_{l|j}) \sum_{u=1}^{\infty} \pi_{u|j,l} \, K(x|\psi_{u|j,l})}{\sum_{h=1}^{\infty} \pi_h \sum_{l=1}^{\infty} \pi_{l|h} \, K(m|x; \omega_{l|h}) \sum_{u=1}^{\infty} \pi_{u|h,l} \, K(x|\psi_{u|h,l})}.$$

- Conditional distribution of M given X:

$$f(m|x) = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} W_{j,l}(x) \cdot K(m|x; \omega_{l|j})$$

where,

$$W_{j,l}(x) = \frac{\pi_j \pi_{l|j} \sum_{u=1}^{\infty} \pi_{u|j,l} \, K(x|\psi_{u|j,l})}{\sum_{h=1}^{\infty} \sum_{g=1}^{\infty} \pi_h \pi_{g|h} \sum_{u=1}^{\infty} \pi_{u|h,g} \, K(x|\psi_{u|h,g})}.$$

Thus, although the local regression models $K(y|m, x; \theta_j)$ and $K(m|x; \omega_{l|j})$ are generalized linear models, the global regression models $f(y|m, x)$ and $f(m|x)$ are **computationally tractable, flexible, non-linear, non-additive models.**

Computation

MCMC: Data, priors on cluster-specific parameters → at each iteration, first update cluster membership (Algorithm 8 of [?]) and then update the cluster-specific parameters → store them

Post-processing computation:

(a) Draw cluster and then draw the covariates

(b) Given the covariates from the previous step in (a), draw an m-subcluster and subsequently draw the mediators in such a way that the q^{th} mediator is induced under the treatment status a_q, for some fixed $\{a_1, a_2, \cdots, a_Q\} \in \{0, 1\}$

(c) Given the values from (a) and (b), compute $E(Y | A = a, L = l, M = m, \theta^*, \omega^*, \psi^*, s)$

(d) Repeat (a)-(c) T times and use MC Integration to compute $E(Y(a, M(a_1, a_2, \cdots, a_Q)))$